We denote by β_l°, $\beta_{j,l}^\circ$ the controls which minimize the functionals (5.2), by $\eta_{j,l}(x)$ the value of the function $\eta_j(x)$ that corresponds to the control $\beta_{j,l}^\circ$, and by $\eta_{j,l}^\circ$ the maximum value of the function $\eta_{j,l}$ on the segment $[0, 1]$. It can be shown that a constant $c_2 > 0$ exists independent of ε such that the inequalities

$$V(\mathring{\beta}_{j,l}) \leqslant V(\beta_0) + c_2\varepsilon^{j-l+1} \tag{5.3}$$

$$\mathring{\eta}_{j,l} \leqslant y^\circ + c_2\varepsilon^{2l/3}, \quad \sup_x y(T, x, \mathring{\beta}_{j,l}) \leqslant y^\circ + c_2\varepsilon^{2l/3} \tag{5.4}$$

are satisfied.

We will select the parameter l from the condition that the degree of error with respect to functional (5.3) and with respect to the maximum value of the deflection (5.4) are equal. Then when $l = 0,6(j+1)$, the optimal control of the set of equations (2.2) that minimizes the quality criterion $V_{j,l}(\beta)$ determines the $\varepsilon^{0,4(j+1)}$-optimal control in the converse problem of rod shape optimization.

In particular, when $j = 2$, the optimal control of the system of ordinary differential equations

$$\frac{d^2y_0}{dx^2} + a\beta y_0 = -a\beta m(x)$$

$$\frac{d^2y_1}{dx^2} + a\beta y_1 = -a\beta(1-\beta)\varphi(\rho(x))(y_0 + m(x)).$$

$$\frac{d^2y_2}{dx^2} + a\beta y_2 = -a\gamma\beta(1-\beta)\left\{y_1 - \beta\varphi(\rho(x))(y_0 + m(x)) \times \int_0^T \varphi(t + \rho(x))[\exp(-\gamma t) - \exp(-\gamma T)]\,dt\right\}$$

is the zero boundary condition that minimizes the functional

$$V(\beta) + \varepsilon^{-4/3}\int_0^1 \max[\eta_2(x) - y^\circ, 0]\,dx$$

$$\eta_2(x) = y_0(x) + \varepsilon(1 - \exp(-\gamma T))\,y_1(x) + \varepsilon^2 y_2(x)$$

The proposed algorithm of optimal shape determination can be applied to rods with other forms of support. It is then only necessary to investigate the supplementary conditions that guarantee that inequality (2.3) is satisfied.

REFERENCES

1. ARUTIUNIAN H.Kh., Some Problems of the Theory of Creep. Moscow-Leningrad, Gostekhizdat, 1952.
2. ARUTIUNIAN H.Kh., On the theory of the creep of non-uniformly heredity-aging media. Dokl. AN SSSR, Vol.229, No.3, 1976.
3. ARUTIUNIAN N.Kh. and KOLMANOVSKII V.B., The optimization problem in the theory of creep of inhomogeneous beams subject to aging. Prikl. Mekhanika, Vol.15, No.9, 1979.
4. VASIL'EV F.P., Methods of Solving Extremal Problems. Moscow, NAUKA, 1981.
5. PONTRIAGIN L.S., BOLTIANSKII V.G., GAMKRELIDZE R.V. and MISHCHENKO E.F., The Mathematical Theory of Optimal Processes. Moscow, NAUKA, 1976.
6. CHERNOUS'KO F.L. and KOLMANOVSKII V.B., Computational and approximate methods of optimal control. In: Results of Science and Technology. Mathematical Analysis. Vol.14, Moscow, VINITI, 1977.

Translated by J.J.D.

PMM U.S.S.R., Vol.48, No.1, pp.40-49, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

RATIONAL SCHEMES FOR REINFORCING LAMINAR PLATES FROM COMPOSITE MATERIALS*

V.M. KARTVELISHVILI and V.V. KOBELEV

New problems for optimizing the internal structure of plates from a laminar composite for a number of local and integral functionals are considered. A model of a laminar-fibrous composite plate is described. Prior to optimization, the plate is a packet of monolayers homogeneous over the thickness. The monolayers are formed by periodic unidirectional stacking of reinforcing fibres in an elastic matrix. To determine the effective elastic properties of the monolayers, a homogenized model of the composite material is used. The concentration of reinforcing fibres or the angles of orientation of the axes of material anisotropy in a given number of

*Prikl.Matem.Mekhan., 48, 1, 68-80, 1984

monolayers are selected as control functions. A constraint, which does not enable one to go beyond the framework of application of the initial model for the composite, is imposed on the gradient of the functions giving the mechanical characteristics of the monolayers. The necessary conditions for optimality are obtained. Successive approximations are used to seek approximate analytic solutions. Optimal schemes are found for reinforcing laminar plates of different configuration in the following problems: minimization of the maximum plate deflection, maximization of their integral stiffness and the first natural vibration frequency.

1. Model of a laminar fibrous composite plate. 1°. *Tensor of the effective stiffness of a packet of monolayers homogeneous over the thickness.* In a rectangular coordinate system $x_1x_2x_3$ we consider an elastic anisotropic plate of constant thickness h. The plane of elastic symmetry of the plate $x_3 = 0$ occupies a domain Ω with boundary Γ.

The plate has a laminar-fibrous structure obtained by regular stacking of N reinforced monolayers in a packet. The number of layers in the packet is fairly large. The homogeneity of the packet over the thickness is assured by periodic stacking of monolayers with identical characteristics along the x_3 axis.

The monolayers have a thin periodic structure. A separate s-th monolayer is formed by a quasiperiodic system (possessing the so-called symmetry of short-range order /1/) of local unidirectional fibres at an angle φ_s, stacked in an elastic anisotropic matrix (here and henceforth $s = 1, \ldots, N$).

Let $D = \{D_{ijkl}(x_1, x_2), i, j, k, l = 1, 2\}$ be the tensor of effective plate stiffnesses /2/ and let us express the components of the tensor $D_{ijkl}(x_1, x_2)$ in terms of components of the monolayer effective stiffness tensor $d_{ijkl}^s(x_1, x_2)$ as well as the angles of reinforcing fibre orientation in the monolayers φ_s. To do this we introduce the rotation matrix $M(\varphi_s) = \{M_{ij}(\varphi_s), i, j = 1, 2\}$, where

$$M_{11}(\varphi_s) = M_{22}(\varphi_s) = \cos\varphi_s, \quad M_{12}(\varphi_s) = -M_{21}(\varphi_s) = \sin\varphi_s$$

In the notation used, we have

$$D_{ijkl}(x_1, x_2) = 12\sum_{s=1}^{N} s^2 G_{ijkl}^s(x_1, x_2) \tag{1.1}$$

$$G_{ijkl}^s = M_{im}(\varphi_s) M_{jn}(\varphi_s) M_{kp}(\varphi_s) M_{lq}(\varphi_s) d_{mnpq}^s$$

For sufficiently large N and assumptions about the homogeneity of the packet of layers with respect to the thickness (i.e., the periodicity of stacking the monolayers with different directions of the axes of anisotropy), we obtain the desired estimate for the components of the packet effective stiffness tensor from (1.1) in the form

$$D_{ijkl}(x_1, x_2) = N^2\sum_{s=1}^{N} G_{ijkl}^s \tag{1.2}$$

2°. *Tensor of effective stiffnesses of a fibrous monolayer.* We will describe the internal structure of the fibre monolayer taken separately. To do this we connect a local orthogonal coordinate system $y_1^s y_2^s$ to the s-th monolayer such that the axis y_1^s makes an angle φ_s with the x_1 axis. We direct the axis y_2^s along the reinforcing fibres. In each layer the fibres with Lamé coefficients λ_1, μ_1 are stacked unidirectionally in the elastic isotropic matrix to which the Lamé coefficients λ_2, μ_2 correspond. The stacking has a thin periodic structure with period ε along the y_1^s axis. The macroscopic stiffness characteristics of the plate are here determined for the limit state as $\varepsilon \to 0$. We select a two-component model of the composite with the following piecewise-constant distribution law for the elastic constants with period ε in the s-th layer, we later omit the superscript s in the local coordinate system notation

$$\begin{aligned} &\lambda_s(y_1, y_2) = \lambda_1, \ \mu_s(y_1, y_2) = \mu_1, \ [y_1/\varepsilon] \leqslant y_1 \leqslant [y_1/\varepsilon] + \varepsilon v_s \\ &\lambda_s(y_1, y_2) = \lambda_2, \ \mu_s(y_1, y_2) = \mu_2, \ [y_1/\varepsilon] + \varepsilon v_s \leqslant y_1 \leqslant [y_1/\varepsilon] + \varepsilon \end{aligned} \tag{1.3}$$

The square brackets in (1.3) denote the integer part of a number. A function giving the macroscopic concentration of the reinforcement stacking in the elastic matrix as the period ε tends to zero is denoted by $v_s = v_s(y_1, y_2)$. From the technological viewpoint, for the model to be applicable it is sufficient that the rate of change of the function $v_s(y_1, y_2)$ should be much less than the characteristic rate of change of the elastic constants of the composite in the period ε as $\varepsilon \to 0$. Consequently, we impose the following constraints on the modulus of the gradient of the function $v_s(y_1, y_2)$:

$$|\nabla v_s(y_1, y_2)| = \sqrt{(v_s)_{,1}^2 + (v_s)_{,2}^2} \leqslant \varkappa, \quad (y_1, y_2) \in \Omega \tag{1.4}$$

Here and henceforth the subscripts after the comma denote partial derivatives with respect to the appropriate coordinate.

We will express the effective stiffness d^s_{ijkl} from (1.2) in terms of $\lambda_1, \mu_1, \lambda_2, \mu_2, v_s(y_1, y_2)$ as $\varepsilon \to 0$ by using the homogenization procedure /3, 4/. For the type of two-component composite (1.3) under consideration, by using the relationships for homogenized models of regularly reinforced composite plates /5/, we obtain after integration with respect to the period ε and allowing ε to tend to zero, taking (1.4) into account,

$$d_1^s = A_0^{-1},\ d_2^s = A_1 A_0^{-1},\ d_3^s = 2\langle\mu\rangle \tag{1.5}$$

$$d_4^s = A_1^2 A_0^{-1} + A_2,\ d_5^s = d_6^s = 0$$

$$A_0 = \langle a\rangle,\ A_1 = \langle b\rangle,\ A_2 = \langle c\rangle \tag{1.6}$$

$$a = (\lambda + 2\mu)^{-1},\ b = \lambda a,\ c = 4\mu(\lambda+\mu)a$$

$$\langle f\rangle = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon f\,dy_1$$

Here and henceforth, for brevity, the 1111, 1122, 1212, 2222, 1112, 1222 are replaced, respectively, by the subscripts 1, 2, 3, 4, 5, 6 (for example, $d^s_{1122} = d_2^s$).

3°. *The technologically acceptable structure of a laminar fibrous composite.* Within the framework of the model being investigated, we will write down and analyse the expressions for the effective stiffness tensors corresponding to technologically acceptable structures of laminar-fibrous plates. Using the expressions obtained, the optimal characteristics, in a specific sense, of these structures will later be sought.

The structure 0 (reference). This structure corresponds to the initial state of the plate prior to optimization. Here the packet of monolayers is homogeneous in thickness, the directions of the reinforcing fibres of all the layers agree: $\varphi_s = \varphi^\circ$, and the macroscopic concentrations are equal and constant: $v_s(x_1, x_2) = v_s(y_1^s, y_2^s) = v^\circ$, $0 \leqslant v_{\min} \leqslant v^\circ \leqslant v_{\max} \leqslant 1$. In this case the fibres in the monolayers form a strictly periodic stack, and we have for the components of the effective stiffness tensor of each monolayer: $d^s_{ijkl}(x_1, x_2) = d_{ijkl}$. Hence, taking (1.2) into account, we obtain that

$$D_{ijkl}(x_1, x_2) = \mathring{D}_{ijkl} \equiv N^3 \mathring{d}_{ijkl} \tag{1.7}$$

where $\mathring{d}_{ijkl}$ are calculated from (1.5) and (1.6) for $v_s = v^\circ$.

Structure 1 (inhomogeneous macroscopic concentration in the monolayer). We assume that there are $n < N$ periodically stacked plates of monolayers over the thickness, with concentrations variable in Ω : $v^1(x_1, x_2)$: $v_k = v^1(x_1, x_2)$, $k = 1, \ldots, n$, $0 \leqslant v_{\min} \leqslant v^1 \leqslant v_{\max} \leqslant 1$. The remaining $N - n$ layers have the reinforcement concentration $v_l = v^\circ$, $l = n+1, \ldots, N$. The directions of the reinforcing fibres agree in the whole packet: $\varphi_s = \varphi^\circ$. Introducing $\beta = n/N$, we obtain

$$D_{ijkl}(x_1, x_2) = (1-\beta)\mathring{D}_{ijkl} + \beta v^1(x_1, x_2) D^1_{ijkl} \tag{1.8}$$

The components D^1_{ijkl} are given by the expressions

$$D_1^1 = -N^3 A_0^{-2}\{a\},\ D_2^1 = N^3 A_0^{-2}(\{b\}A_0 - \{a\}A_1) \tag{1.9}$$

$$D_4^1 = N^3 A_1 A_0^{-2}(\{b\}A_0 - \{a\}A_1) + \{c\}N^3$$

$$D_3^1 = 2N^3\{\mu\},\ D_5^1 = D_6^1 = 0$$

where $\{f\} = \lim_{\delta\to 0}(f([y_1/\varepsilon] + \varepsilon v_s - \delta) - f[y_1/\varepsilon] + \varepsilon v_s + \delta))$ is the jump in the functions f on the fibre-matrix interface.

Structure 2 (variable orientation of anisotropy axes in the layer). We now assume that the reinforcing fibre concentrations v_s in all the monolayers are identical and equal to v°, i.e., $v_s = v^\circ$. Let $m < N$ monolayers be reinforced in the direction α: $\varphi_l = d(x_1, x_2)$, $l = 1, \ldots, m$, $\alpha \neq \varphi^\circ$, where φ° is the reinforcing direction of the remaining $N - m$ layers in the packet. Introducing the quantity $\gamma = m/N$, we have

$$D_{ijkl} = (1-\gamma)\mathring{D}_{ijkl} + \gamma D^2_{ijkl} \tag{1.10}$$

where the components D^2_{ijkl} are given by the expressions ($s = \sin\alpha$, $c = \cos\alpha$)

$$D_1^2 = U_1c^4 + U_2s^4 + U_3,\ D_4^2 = U_1s^4 + U_2c^4 + U_3,$$

$$D_2^2 = {}^1\!/_4(U_1 + U_2)s^2c^2 + D_2^\circ,\ D_3^2 = {}^1\!/_2(U_1 + U_2)s^2c^2 + D_3^\circ$$

$$D_5^2 = {}^1\!/_2 sc(U_2s^2 - U_1c^2),\ D_6^2 = {}^1\!/_2(U_2c^2 - U_1s^2)sc$$

$$U_1 = D_1^\circ - D_2^\circ - 2D_3^\circ,\ U_2 = D_4^\circ - D_2^\circ - 2D_3^\circ,\ U_3 = D_2^\circ + 2D_3^\circ$$

Hypotheses of short-range order symmetry (on which the introduction of effective moduli is indeed based), formalized for concentrations in the form of the inequalities (1.4), are formalized for angles of orientation $\alpha(x_1, x_2)$ in the form $|\nabla\alpha| = \sqrt{\alpha_{,1}^2 + \alpha_{,2}^2} \leqslant \sigma$, where σ is the characteristic dimension of the short-range order domain ($\sigma \sim \varkappa$). Note that by using the Cauchy inequality the constraint mentioned is reduced to a constraint on the curvature of the stacking lines of the reinforcement (t is a parameter along the stacking trajectory)

$$|\partial\alpha/\partial t| \leqslant |\nabla\alpha| \leqslant \sigma \tag{1.11}$$

2. Formulation of optimization problems. Necessary conditions for optimality, *1°. Formulation of plate structure optimization problems.* Having constructed the model of an anisotropic laminar-fibrous plate in the preceding section, we will now formulate optimization problems for the internal structure of bent and oscillating plates.

Among the constraints imposed on a design in addition to basic requirements in problems of designing composite plates are

$$J_0 = \iint_\Omega h\rho(x_1, x_2)\,d\Omega \tag{2.1}$$

$$J_1 = {}^1/_2 \iint_\Omega f(w, w)\,d\Omega, \quad f(w, u) = D_{ijkl} w_{,ij} u_{,kl} \tag{2.2}$$

$$J_2 = \theta^2 = \min_w \Big(\iint_\Omega f(w, w)\,d\Omega \Big/ \iint_\Omega w^2 h\rho\,d\Omega\Big) \tag{2.3}$$

$$J_3 = \max_{(x_1, x_2)\in\Omega} |w(x_1, x_2)| \tag{2.4}$$

on the total mass of material (2.1) ($\rho(x_1, x_2)$ is the macroscopic density of the composite material), the integral stiffness (2.2), the first natural vibration frequency (2.3), and the maximum deflection (2.4). Here $f(w, u)$ is a symmetric, continuous and positive-definite bilinear form generated by a fourth-order differential operator

$$L[D]\,w \equiv (D_{ijkl} w_{,ij})_{,kl}$$

that is equivalent on the left side of the plate bending differential equation to the transverse forces $q(x_1, x_2)$

$$L[D]\,w = q(x_1, x_2) \tag{2.5}$$

The form of $f(w, u)$ is determined in the set H_A obtained by closure of the set $C^\infty(\Omega)$ of infinitely differentiable functions in $\bar\Omega$ that satisfy the appropriate conditions of stiff clamping and free support in the Sobolev space $W_2^2(\Omega)$

$$(K_j[w])_\Gamma = 0$$

where K_j are linear homogeneous differential operators evaluated on the piecewise-smooth boundary Γ. Fro the two-component model of the composite introduced, the density $\rho(x_1, x_2)$ in (2.1) and (2.3) has the form

$$\rho(x_1, x_2) = \rho_2 + (\rho_1 - \rho_2)\frac{1}{N}\sum_{s=1}^{N} v_s(x_1, x_2)$$

where ρ_2 is the density of the matrix material, and ρ_1 is the density of the reinforcing fibres.

We henceforth assume that the mass of the composite is fixed

$$J_0 = M_0 \tag{2.6}$$

This means that during optimization, condition (2.6) occurs in the set of isoperimetric conditions.

We select as optimizing functional one of the functions (2.2)–(2.4). In other words, for a given mass we will increase the stiffness or the least eigen frequency of the plate.

The concentrations $v_l = v^l(x_1, x_2)$ $(l = 1, \ldots, n)$ which are subject to the technological constraints (1.4) and $0 \leqslant v_{\min} \leqslant v^l \leqslant v_{\max} \leqslant 1$, are the control functions for structure 1. Introducing the new control function $V = v^l - (v_{\max} + v_{\min})/2$ we reduce the last constraint to an inequality more convenient for further examination

$$|V(x_1, x_2)| \leqslant \eta, \quad \eta = (v_{\max} - v_{\min})/2 \tag{2.7}$$

and the isoperimetric condition (2.6) to the equality

$$\psi_1 = \iint_\Omega V(x_1, x_2)\,d\Omega = 0 \tag{2.8}$$

We select the angles of fibre rotation in the monolayers $\varphi_l = \alpha(x_1, x_2)$ $(l = 1, \ldots, m)$ as control functions for structure 2.

We thus formulate the optimization problems for structures 1 and 2.

Structure 1. Find the optimal concentration distribution $V_{\mathrm{opt}}(x_1, x_2) = V(x_1, x_2)$ in Ω which will minimize (2.4) (problem 1), maximize (2.3) (problem 2) under the isoperimetric conditions (1.4) and (2.6) and the constraints (2.7).

Structure 2. Find the optimal angles of rotation $\alpha_{\rm opt}(x_1, x_2)$ in Ω which will maximize (2.2) (problem 3) or (2.3) (problem 4) under the isoperimetric conditions (1.11) and (2.6).

$2^{\rm O}$. *Successive approximations. Necessary optimality conditions.* We will obtain the solution of the above problems by the method of successive approximations, according to which a correction $\delta V^k(x_1, x_2)$, assuring improvement in the structure in the sense required, can be constructed by relying on knowledge of the running structure of the plate being designed at step k. Furthermore, where possible the superscript k will be omitted. Writing down the successive approximations relations, we deduce the necessary optimality conditions from them as a particular limit case in problems where the concentration of inclusions (structure 1) is the control function. To this end, we replace the local constraints (1.4), (2.7) and the local functional (2.4) by their approximate integral analogs.

$$\psi_2 = \left(\frac{1}{S}\iint_\Omega \left|\frac{\nabla V}{\nu}\right|^q d\Omega\right)^{1/q} \leqslant 1, \quad \psi_3 = \left(\frac{1}{S}\iint_\Omega \left|\frac{V}{\eta}\right|^p d\Omega\right)^{1/p} \leqslant 1 \tag{2.9}$$

$$J_3 \approx I_r = \left(\frac{1}{S}\iint_\Omega |w|^r d\Omega\right)^{1/r} \equiv \|w\|_r, \quad S = \iint_\Omega d\Omega \tag{2.10}$$

For p, q, r tending to infinity, (2.9) and (2.10) become (1.4), (2.7), and (2.4).

Considering δV as a vector in infinite-dimensional Hilbert space with the scalar product

$$(a, b) = \iint_\Omega ab\, d\Omega$$

we express the total variation of the functionals being optimized J_n $(n = 1, 2, 3)$ in the step k from (2.2), (2.3), (2.10) and the variations of the constraints ψ_m $(m = 1, 2, 3)$ from (2.8) and (2.9) in terms of the variation of the control function δV. To eliminate the variation in the phase variable δw we introduce the conjugate variable to $u(x_1, x_2)$, equal to $u = -w$ for the functional (2.2), $u = -2w$ for the functional (2.3), or subject to the equation

$$(D_{ijkl}u_{,kl})_{,ij} + S^{-1}(w/\|w\|_r)^{r-1} = 0$$

for the second functional (2.9). Then the variations of the optimizable functionals and constraints take the form

$$\delta J_n = (\Phi_n, \delta V), \ \delta\psi_n = (\Psi_n, \delta V); \ n = 1, 2, 3 \tag{2.11}$$

$$\Phi_1 = {}^1/_2 D_{ijkl}{}^1 w_{,ij} w_{,kl}, \quad \Phi_2 = 2\Phi_1 + \theta^2(\rho_1 - \rho_2) h w^2 \tag{2.12}$$

$$\Phi_3 = D^1_{ijkl} w_{,ij} u_{,kl}$$

$$\Psi_1 = 1, \quad \Psi_2 = \frac{\nabla(|\nabla V|^{p-2}\nabla V)}{\varkappa S\|\nabla V\|_p^{p-1}}, \quad \Psi_3 = \frac{|V|^{q-1}}{\eta S\|V\|_q^{\ q}} \tag{2.13}$$

The constraints (2.8) and (2.9) govern the allowable domain Λ of the space of controls whose boundary $\partial\Lambda$ is given if there are equality signs in (2.9). The form of the expression for the improving corrections $\delta V_m{}^k$ depend in each step on the number and form of the active constraints being taken into account (m is the number of constraints taken into account).

The control function $V^k = V^{k-1} + \delta V_m{}^k$, as well as the corresponding phase and conjugate variables w^k, u^k satisfy here the optimality conditions if the variations calculated in this manner vanish when substituted into the expression for the variations of the control functions.

For instance, if the control vector belongs to the interior of the domain Λ (i.e., $V^{k-1} \subset \Lambda \setminus \partial\Lambda$), then a variation of the form

$$\delta V_0{}^k = \pm\tau\Phi_n \quad (0 < \tau \ll 1) \tag{2.14}$$

can, in particular, be the allowable improving correction $\delta V_0{}^k$, where the plus sign corresponds to maximization of the functional. Therefore, the optimality condition for the problem without constraints has the form $\Phi_n = 0$.

If the control vector belongs to the boundary of the domain $\partial\Lambda$ (we note that the isoperimetric condition of constancy of the mass $\psi_1 = 0$ should be satisfied at each step), while the projection of the vector $\delta V_0{}^k$ is positive on the external normal to the surface $\partial\Lambda$, then the allowable improving variation is constructed by successively taking into account the constraints $(\Psi_m, \delta V_m{}^k) = 0$ $(m = 1, 2, 3)$ by using the Gramm-Schmidt orthogonalization procedure

$$\delta V_m{}^k = \delta V^k_{m-1} - (\delta V^k_{m-1}, \Psi_m)\Psi_m/(\Psi_m, \Psi_m) \tag{2.15}$$

(we recall that the subscript m denotes the number of constraints taken into account). Consequently, the improving variation in the problem where just the isoperimetric condition of constancy of the mass of the composite material is taken into account, has the form $\delta V_1^k = \pm\tau\,[\Phi_n - S^{-1}(\Phi_n, 1)]$ and the optimality condition is $\Phi_n = S^{-1}(\Phi_n, 1)$, which converges, say, to the condition of constancy of the strain potential energy density for Φ_1.

A stepwise algorithm for solving optimization problems can be constructed on the basis of the relationships obtained. At each step the phase and conjugate variables are sought successively for the running value of the control functions, and when they are known, we construct the improving correction that assures optimization of the quality functional by taking account of the constraints. If the boundary conditions and the nature of the load enable analytic expressions to be obtained at the running step for the phase and conjuate variables, then the improving correction to the control function can be determined analytically. Analytic solutions of optimization problems for the first approximation $(k = 1)$ will be constructed below.

3. Analytic solutions in problems of selecting the optimal distribution of the reinforcing fibre concentrations. 1°. *Minimization of the maximum deflection of a multilayer plate for a class of forces realizing a normal load.* Consider a rectangular laminar-fibrous plate $\Omega\,\{0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant b\}$ simply supported at the contour Γ and subjected to transverse forces $q(x_1, x_2)$. In this case, the equation for the phase variable $w(x_1, x_2)$ has the form (2.5), and the boundary conditions are written as follows:

$$\begin{aligned} &w(0, x_2) = w(a, x_2) = (D_1 w_{,11} + D_2 w_{,22} + 2D_5 w_{,12})_{\substack{x_1=0\\x_1=a}} = 0 \\ &w(x_1, 0) = w(x_1, b) = (D_2 w_{,11} + D_4 w_{,22} + 2D_6 w_{,12})_{\substack{x_2=0\\x_2=b}} = 0 \end{aligned} \tag{3.1}$$

We consider the function $q(x_1, x_2)$ to belong to the class of functions Q that are summable in the domain Ω and have generalized derivatives with respect to x_1 and x_2 to the required order. For simplicity, we consider the function $q(x_1, x_2)$ to be expanded in the domain Ω in a Fourier series containing only products of the sines

$$\begin{aligned} &q = \sum_{t,s=1}^{\infty} a_{ts} \sin t\bar{x}_1 \sin s\bar{x}_2 \\ &a_{ts} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi q \sin t\bar{x}_1 \sin s\bar{x}_2 \, d\bar{x}_1\, d\bar{x}_2 \\ &\bar{x}_1 = \pi x_1/a, \quad \bar{x}_2 = \pi x_2/b \end{aligned} \tag{3.2}$$

According to the method described in Sect.2, a first approximation is successfully constructed analytically in the problem of minimizing the maximum deflection for the boundary conditions and kind of load mentioned because of selection of the optimal distribution of the reinforcing material concentration (problem 1). Thus, Eqs.(2.11) for the first approximation become

$$\begin{aligned} &\overset{\circ}{D}_{ijkl} \overset{\circ}{w}_{,ijkl} = q, \quad \Phi_3 = D_{ijkl} \overset{\circ}{w}_{,ij} \overset{\circ}{u}_{,kl} \\ &\overset{\circ}{D}_{ijkl} \overset{\circ}{u}_{,ijkl} = -(w/\|w\|_r)^{r-1} S^{-1} \end{aligned} \tag{3.3}$$

The solution of boundary value problems for the phase and conjugate variables w° and u° is found in the form of double Fourier series. Taking account of the nature of (3.3), Green's function can be used to find the functions mentioned. Omitting the lengthy computations, we write the expression for the function $\Phi_3(x_1, x_2)$ from (2.12):

$$\begin{aligned} \Phi_3 = &-4\pi^{-2}\|w^\circ\|_r^{1-r} \int_0^\pi \int_0^\pi \Big\{ (w^\circ)^{r-1} \sum_{i,j,k,l=1}^{\infty} \frac{a_{ij} \sin k\bar{z}_1 \sin l\bar{z}_2}{Q_{ij} Q_{kl}} \times \\ &\sin i\bar{x}_1 \sin j\bar{x}_2 \sin k\bar{x}_1 \sin l\bar{x}_2 \Big\{ \Big[D_1^1 \Big(\frac{i\pi}{a}\Big)^2 + D_2^1 \Big(\frac{j\pi}{b}\Big)^2 \Big] \times \\ &\Big(\frac{k\pi}{a}\Big)^2 + \Big[D_4^1 \Big(\frac{j\pi}{b}\Big)^2 + D_2^1 \Big(\frac{i\pi}{a}\Big)^2 \Big] \Big(\frac{l\pi}{b}\Big)^2 \Big\} + \\ &4D_3^1 \pi^4 \frac{ijkl}{a^2 b^2} \cos i\bar{x}_1 \cos j\bar{x}_2 \cos k\bar{x}_1 \cos l\bar{x}_2 \Big\} \, d\bar{z}_1\, d\bar{z}_2 \\ Q_{mn} = &\, D_1^\circ \frac{m^4 \pi^4}{a^4} + 2(D_2^\circ + 2D_3^\circ) \Big(\frac{mn\pi^2}{ab}\Big)^2 + D_4^\circ \frac{n^4 \pi^4}{b^4} \\ \bar{z}_1 = &\, \pi\zeta_1/a, \quad \bar{z}_2 = \pi\zeta_2/b \end{aligned} \tag{3.4}$$

Let us take account of the isoperimetric condition of constancy of the mass of the composite (2.8) and the constraints (2.9). If just the constrains on the gradient of the function $V(x_1, x_2)$ from the first condition (2.9) is considered, then the vector δV_r^1 from (2.15) is found from (2.14), (2.15)

$$\delta V_0^1 = \Phi_3, \delta V_1^1 = \Phi_3 - S^{-1}(\Phi_3, 1)$$
$$\delta V_2^1 = \delta V_1^1 - \frac{(\delta V_1^1, \nabla [|\nabla V^\circ|^{q-2} \nabla V^\circ]) \nabla (|\nabla V^\circ|^{q-2} \nabla V^\circ)}{(\nabla (|\nabla V^\circ|^{q-2} \nabla V^\circ), \nabla (|\nabla V^\circ|^{q-2} \nabla V^\circ))} \tag{3.5}$$

Taking account of the constraints (2.8) and the second condition (2.9), results according to (2.14) and (2.15), in the improving correction

$$\delta V_2^1 = \delta V_1^1 - \frac{(\delta V_1^1, |V^\circ|^{p-1})}{(|V^\circ|^{p-1}, |V^\circ|^{p-1})} |V^\circ|^{p-1} \tag{3.6}$$

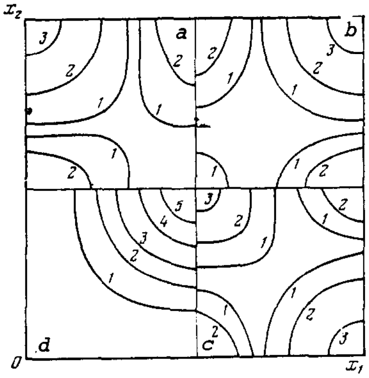

Fig.1

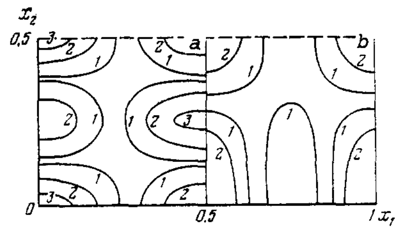

Fig.2

As an illustration, we will select the transverse load $q = \sin \pi x_1 \times \sin \pi x_2$ $(a = b = 1)$. Having determined the improving correction $\delta V = \delta v^1(x_1, x_2)$, we trace the nature of the change in the optimal distributions of the reinforcing fibre concentration v^1 in $n < N$ layers with the growth in the initial (reference) concentration v° of the whole packet. The solutions obtained are symmetric about the lines $x_1 = 0, x_2 = 0.5$. Isolines of the desired optimal distributions are displayed in square quadrants *a, b, c, d* (clockwise) in Fig.1 for $v^\circ = 0, 0.33, 0.66, 1$, respectively. A large concentration of reinforcing material corresponds to isolines with a high order number. Note that if the first constraint in (2.9) is not taken into account, then as *p* and *r* grow it follows from the second constraint of (2.9) and (2.10) that the nature of the optimal distributions tends to the Rayleigh mode. Isolines of the optimal distributions are constructed for $v^\circ = 0.5$ in square quadrants *a* and *b* (left to right) in Fig.2 for $q = \sin 2\pi x_1 \sin 3\pi x_2$ and $q = \sin 3\pi x_1 \sin 2\pi x_2$, respectively $(a = b = 1)$ The asymmetry obtained is due to the anisotropy of the plate stiffness characteristics for the reference concentration distribution.

2°. *Optimization of the eigenfrequencies of a multilayer rectangular plate.* Let us consider the problem of optimizing the first eigenfrequency of a simply supported rectangular plate (problem 2). The frequency of the fundamental mode of the transverse vibrations of a homogeneous anisotropic plate equals /2/ $\theta = \sqrt{h\rho Q_{11}}$, while the first eigenfunction is $w^\circ = \sin x_1 \sin x_2$. Performing calculations similar to those carried out for the stiffness optimization problems, we obtain an expression for the function $\Phi_2(x_1, x_2)$ from (2.12) in the form

$$\Phi_2 = 4\pi^4 a^{-1} b^{-1} [(D_1^1 a^{-4} + 2D_2^1 b^{-2} a^{-2} + D_4^1 b^{-4}) \sin^2 x_1 \sin^2 x_2 + 4a^{-2} b^{-2} D_3^1 \cos^2 x_1 \cos^2 x_2]$$

It is seen that the formula obtained agrees with Φ_3 from (3.4), constructed for the load $q = \sin x_1 \cdot \sin x_2$ apart from a constant. Here it is necessary to set $r = 2$ in the expression for Φ_3.

The property mentioned is even valid for high frequencies, namely, the pattern of the concentration distribution in the problem of optimizing the eigenfrequencies corresponding to the vibrations mode $w^\circ = \sin sx_1 \sin tx_2$ agrees with the concentration distribution in the stiffness optimization problem for the transverse load $q = \sin sx_1 \sin tx_2$.

Isolines of the optimal distributions of bonding material are constructed in Figs.1 and 2, respectively, for $s = t = 1$ (Fig.1); $s = 2, t = 3$ (Fig.2*a*); $s = 3, t = 2$ (Fig.2*b*). Values of the initial reference concentrations are mentioned in Sect.1°.

3°. *Optimation of the overall stiffness of a multilayer elliptical plate subjected to a normal load.* We will present the solution of the problem of maximizing the overall stiffness. Consider an elliptical plate clamped rigidly along the edge and bent by the normal load

$q(x_1, x_2) = q_0$. We direct the axes x_1 and x_2 along the principal axes of the ellipse and denote the magnitudes of the principal semi-axes by a and b. The boundary conditions of the problem corresponding to rigid clamping have the form

$$w = 0, \quad \frac{dw}{dn} = \frac{\partial w}{\partial x_1}\cos(n, x_1) + \frac{\partial w}{\partial x_2}\cos(n, x_2) = 0 \tag{3.7}$$

The fourth-degree polynomial /2/

$$w^\circ = (q_0 a^4/8D')(1 - x_1^2/a^2 - x_2^2/b^2)^2 \tag{3.8}$$
$$D' = 3D_1^\circ + 2c^2(D_2^\circ + 2D_3^\circ) + 3D_4^\circ c^4, \quad c = a/b$$

is a solution of the bending equation of an anisotropic plate with coefficients D_{ijkl} and boundary conditions (3.7) in the elliptical domain $\Omega\,\{x_1^2/a^2 + x_2^2/b^2 \leqslant 1\}$.

Here the function $\Phi_1(x_1, x_2)$ from (2.12) equals

$$\Phi_1 = D_1^1(3x_1^2a^{-4} + x_2^2a^{-2}b^{-2} - a^{-2})^2 + 8(D_2^1 + 2D_3^1)x_1^2x_2^2a^{-2}b^{-2} + D_4^1(3x_2^2b^{-4} + x_1^2a^{-2}b^{-2} - b^{-2})^2$$

The optimal concentration distributions yields (3.5), (3.6), where Φ_3 must be replaced by Φ_1.

4. Rational schemes for fibre orientation in monolayers. *1°. Formulation of the problem.* Let us consider the problem of the analytical determination of rational schemes for orientation of the reinforcing fibres in flexible laminar plates. We select the magnitude of the strain energy (integral stiffness) (2.2) of the plate as the function to be optimized and we consider the problem of minimizing this functional by a rational selection of the orientation of the lines of anisotropy (problem 3 for structure 2).

Let us associate a Cartesian x_1x_2 system of coordinates with the middle plane of the plate. The plate deflections $w(x_1, x_2)$ due to the action of transverse loads are considered to be small. At each point the bending of the middle surface is characterized by the principal radii of curvature of the bent surface k_1, k_2, which are calculated as the roots of the quadratic equation /6/

$$(w_{,11}w_{,22} - w_{,12}^2) - (w_{,22} + w_{,11})k + k^2 = 0 \tag{4.1}$$

The equations of the lines of curvature $X_1 = X_1(x_1, x_2)$, $X_2 = X_2(x_1, x_2)$ are integral curves of the differential equations

$$X_{1,1} = p + \sqrt{p^2+1}, \quad X_{2,1} = p - \sqrt{p^2+1}, \quad p = (w_{,22} - w_{,11})/2w_{,12} \tag{4.2}$$

If the bent surface $w(x_1, x_2)$ does not contain umbilical points (round-off points at which $k_1 = k_2$), then there are two orthogonal families of lines of curvature X_1, X_2. We use the families $X_1(x_1, x_2)$ and $X_2(x_1, x_2)$ mentioned as a new curvilinear coordinate system. The field of local angles of rotation of the axes (X_1X_2) relative to (x_1x_2) is denoted by $\xi = \xi(x_1, x_2)$. Note that the orthogonality of the coordinate lines can be retained at the umbilical points by selecting the two directions that are a continuation of the lines of curvature in the neighbourhood of the umbilical point, as the principal directions.

Let $\zeta = \zeta(x_1, x_2)$ be the angle between the axis X_2 and the axis y_2 that coincides with the direction of the reinforcing fibres in the initial structure. Then the condition $\alpha = \xi + \zeta$ holds for the desired angle of fibre stacking α with respect to the axis x_2 of the Cartesian system of coordinates. Therefore, the problem will be solved if the angles $\xi(x_1, x_2)$ and $\zeta(x_1, x_2)$ are found.

2°. Rational methods of reinforcing with small curvature of the reinforcement stacking lines. We use successive approximations to find the angles ξ and ζ : by knowing the value of the deflection function w in the step k and, therefore, knowing the distribution of the angles ξ and the curvature k_1, k_2 as well as the running distribution ζ, we find a new rational distribution of the angles $\zeta(x_1, x_2)$ which improves the functional (2.2) being optimized. We hence require that the change in the distribution of the deflections w due to the variation $\delta\zeta$ in the desired function ζ be sufficiently small at each step. According to (1.10), this can be achieved if at each step

$$D_{ijkl}^\circ w_{,ij}w_{,kl} \gg \gamma(D_{ijkl}^2 - D_{ijkl}^\circ)w_{,ij}w_{,kl}$$

The inequality will be satisfied in two cases: a) if the step-by-step change in the angle $\delta\zeta$ being optimized is insignificant ($\delta\zeta \ll 1$); b) the number of layers m in which the orientation of the angle ζ (or α) is selected is small compared with the total number of layers N.

Using the above method, we construct first the improving variations of the function ζ for the first case ($\delta\zeta \ll 1$). The function $f(w, w)$ from (2.2) that is dependent on the principal curvatures k_1, k_2 and other constants of the packet is written in the following form in the X_1X_2 system of coordinates

$$f = {}^1/_8 f' = {}^1/_8 (f_0 + f_1 \cos 2\zeta + f_2 \cos^2 2\zeta), \quad f_0 = k_1^2 (Q_1 - Q_3) + 2k_1 k_2 (Q_3 + Q_4) + k_2^2 (Q_1 - Q_3) \tag{4.3}$$
$$f_1 = Q_2 (k_1^2 - k_2^2), \quad f_2 = 2Q_3 (k_1 - k_2)^2, \quad Q_1 = 3D_1^\circ - 3D_4^\circ + 2D_2^\circ + 4D_3^\circ, \quad Q_2 = D_1^\circ - D_4^\circ$$
$$Q_3 = D_1^\circ + D_4^\circ - 2D_2^\circ - 4D_3^\circ, \quad Q_4 = D_1^\circ + D_4^\circ + 6D_2^\circ - 4D_3^\circ$$

The problem of maximizing the overall stiffness (minimization of the compliance)

$$\iint_\Omega qw \, d\Omega \to \min_\zeta$$

by using the variational principle

$$\Pi^* = \min_w \Pi = \min_w \left(J_1 - \iint_\Omega qw \, d\Omega \right)$$

reduces to solving a maximin problem /7/.

In fact, because $\Pi^* = -J_1$, we have $\min_\zeta J_1 = \min_\zeta (-\Pi^*)$. The expression for the variations of the functional J_1 in terms of the variation of the control function $\delta\zeta$ in Π^* written in the form

$$\delta J_1 = (\Phi, \delta\zeta), \quad \Phi = {}^1/_4 \sin 2\zeta \, (f_1 + 2f_2 \cos 2\zeta) \tag{4.4}$$

The variation of the approximate integral analog of the constraint (1.11), constructed as in the first condition of (2.9), has the form

$$\delta\psi = (\Psi, \delta\zeta), \quad \Psi = (\sigma S \, \| \nabla\zeta \|_p^{p-1})^{-1} \nabla (| \nabla\zeta |^{p-2} \nabla\zeta) \tag{4.5}$$

By using the expressions obtained and the algorithm described, an approximate solution of the optimization problem is constructed. The step-by-step change in the angle being optimized (the improving variation) is given by the formula

$$\delta\zeta = -\tau \left(\Phi - \Psi \iint_\Omega \Phi\Psi \, d\Omega \Big/ \iint_\Omega \Psi^2 \, d\Omega \right)$$

We will present some examples of the utilization of the formulas obtained. We consider first a simply supported square plate $(0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1)$, subjected to a distributed load of the form $p = q \sin \pi x_1 \sin \pi x_2$. In this case the deflection function $w(x_1, x_2)$ is proportional to the load distribution while the lines of curvature of the bent plate surface are a family of lines parallel to the diagonals of the square. Initially (in the zeroth step), the reinforcement is laid out parallel to the x_2 axis, and consequently, makes a constant angle $\zeta = \pi/4$ with the lines of curvature. Using the expression for the principal curvatures

$$k_i = \sin \pi x_1 \sin \pi x_2 \pm | \cos \pi x_1 \cos \pi x_2 |, \quad i = 1, 2$$

and taking account of (4.4) and (4.5), we obtain

$$\delta\zeta = \tau \sin 2\pi x_1 \sin 2\pi x_2 \, \mathrm{sign} (\cos \pi x_1 \cos \pi x_2)$$

where $\tau > 0$ is a small constant. The function $\delta\zeta(x_1, x_2)$ is positive for all points of the plate. Therefore, the tendency is manifest for the formation of a slope at a definite side of the reinforcement stacking lines over the whole plane of the plate (the solid lines in Fig.3). We note that an exact solution of the optimization problem exists for the above-mentioned of loading a square plate: the reinforcing lines are parallel to the diagonals of the square, hence the maximum deflections are optimal and the reference plates are cited as $(D_1 + D_4 + 2D_2 + 4D_3)/(4D_1)$, which is 0.35 for a graphite-epoxy composite and 0.3 for a boron-epoxy composite. In addition to the rational stacking (the solid lines) in Fig.3, the optimal scheme (the dashed lines) is also displayed for comparison. It is seen that the approximate method of constructing a rational scheme manifests the characteristic features of the optimal solution.

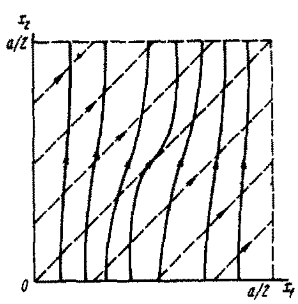

Fig.3

Let us consider the problem of determining a rational scheme for reinforcing a stiffly clamped circular plate loaded by a uniform normal load. The plate deflection is given by (3.8), the principal curvatures of the bent surface equal $k_1 = c_0 (x_1^2 + x_2^2 - 1)$, $k_2 = (3x_1^2 + 3x_2^2 - 1) c_0$, $c_0 = qa^4/8D'$. In the first step the reinforcement stacking lines make an angle $\zeta = \mathrm{arctg}(x_1/x_2)$ with the lines of curvature. Substituting the formula mentioned into (4.4) and (4.5), we obtain the desired function

$$\delta\zeta = \tau x_1 x_2 [Q_2 (2r^2 - 1) + 4Q_3 (x_2^2 - x_1^2)], \quad r = \sqrt{x_1^2 + x_2^2}$$

3°. *Rational methods of reinforcing with arbitrary curvature of the reinforcement stacking lines.* We will examine still another possible method for making a rational selection of the fibre stacking direction, i.e., the case when the number of layers in which the orientation of the reinforcing material is selected is small compared with the total number of layers in the packet. We assume that $\sigma \to \infty$ in (4.5). This means that large curvatures of the reinforcing fibre stacking lines are allowed in (1.11). Making this simplifying assumption, we separate the energy integral into two parts

$$J_1 = \frac{1-\gamma}{2} \iint_{\Omega} D_{ijkl}(\varphi_0)\, w_{,ij} w_{,kl}\, d\Omega + \frac{\gamma}{2} \iint_{\Omega} D_{ijkl}(\alpha)\, w_{,ij} w_{,kl}\, d\Omega \tag{4.6}$$

The problem of determining a rational scheme reduces now to determining $\alpha(x_1, x_2)$ and $\zeta(x_1, x_2)$ from the condition of the minimum of the second component in (4.6). Because of the smallness of γ the distribution $w(x_1, x_2)$ is here considered fixed at each step.

The extremum is reached in (4.6) if the fibre direction and the lines of curvature either agree, or are mutually perpendicular, or make an angle $\zeta = {}^1/_2 \arccos(f_1/(2f_2))$. In particular, the optimum is achieved if the direction of the stacking lines agrees with the direction of the lines of maximum curvature. The values of the second component in (4.6) for the three regimes mentioned are, respectively, $f_0 + f_1 + f_2$, $f_0 - f_1 + f_2$, and $f_0 - f_1^2/(4f_2)$.

Note that this method of selecting the fibre orientation results in a gain in the functional (4.6) provided that $k_1 \neq k_2$. At the umbilical points $(k_1 = k_2)$ the direction of fibre orientation is selected so as to ensure smoothness of the reinforcement stacking lines. This method of selecting the stacking orientation of the reinforcing fibres is suitable even in the case when σ is finite. The domains in which the curvature of the stacking lines exceeds the ultimately admissible value of σ are placed arbitrarily (in particular, with the condition for ensuring smoothness of the reinforcing lines). However, even in this case this method of rational orientation improves the quality functional, as can be shown by using energy estimates.

We emphasize that if the quantity γ is small and the distribution of the deflections in each step depends substantially on the desired fibre orientation $\alpha(x_1, x_2)$, the problem can be solved, using the necessary optimality conditions /8/, by numerical methods.

Let us construct rational stacking schemes using the algorithm proposed. We consider a simply supported square plate, loaded as in Sect.3 by a sinusoidal load. The reinforcement stacking lines either agree with the lines of curvature (the family of lines parallel to the diagonals of the square), or make an angle $\zeta = {}^1/_2 \arccos(-Q_2 \operatorname{tg} \pi x_1 \operatorname{tg} \pi x_2 / 2Q_3)$. The latter reinforcing method does not result in a gain in the functional in this case, while the first two yield an identical value of the gain. We note that the rational reinforcing scheme in the example presented agrees completely with the optimal scheme (item 2° of this section).

The reinforcing method in which the fibres are also laid out along the lines of maximum curvature results in a gain in the functional for a circular plate. This method governs the rational reinforcing scheme in which the fibres are laid out in an annular direction in a circle of radius $1/\sqrt{2}$, and in a radial direction in the rest of the plate.

A model of layered fibre-bonded composite plates has been constructed above, which is based on the hypothesis of a monolayer packet homogeneous over the thickness, and on an analysis of a fibre monolayer by using the homogenization procedure. Taking account of the constraints on the gradients of the functions yielding the concentration and stacking angle of the reinforcing fibres is sufficient for the model used to be applicable. Within the framework of the model under consideration, optimization problems are posed, and optimality conditions are written down. We note that utilization of constraints on the concentration gradient governs the existence of the classical solution of the optimization problem /9/. The method of constructing approximate solutions enables an analytical determination to be made of rational reinforcing schemes for elements of composite structures by relying on a knowledge of the known solutions for the states of stress and strain of homogeneous plates with rectilinear anisotropy.

REFERENCES

1. SIVUKHIN D.V., General Physics Course. Vol.2, Thermodynamics and Molecular Physics, NAUKA, Moscow, 1975.
2. LEKHNITSKII S.G., Anisotropic Plates. Gostekhizdat, Moscow-Leningrad, 1947.
3. BENSOUSAN A., LIONS J.-L. and PAPANICOLAOU G., Asymptotic Analysis for Periodic Structures. North-Holland, Amsterdam, 1978.
4. KOZLOV S.M., Averaging differential operators with almost-periodic rapidly oscillating coefficients, Matem. Sb., Vol.107, No.2, 1978.
5. DUVAUT J., Functional analysis and mechanics of a continuous medium, Theoretical and Applied Mechanics. Trans. Fourteenth Intern. Congr. IUTAM, Mir, Moscow, 1979.
6. DUBROVIN B.A., NOVIKOV S.P. and FOMENKO A.T., Modern Geometry, NAUKA, Moscow, 1979.
7. BANICHUK N.V., Optimization of the Shape of Elastic Solids, NAUKA, Moscow, 1980.
8. LURIE K.A., CHERKAEV A.V. and FEDOROV A.V., Regularization of optimal design problems for bars and plates. Pt. 1-2, J. Optimiz. Theory and Appl., Vol.37, No.4, 1982.
9. LITVINOV V.G., Problem of optimal control of the natural frequency of a plate of variable thickness, Zh. Vychisl. Matem. Matem. Fiz., Vol.19, No.4, 1979.

Translated by M.D.F.